Approximated by finite-dimensional homomorphisms into Simple C*-Algebras with Tracial Rank One

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Abstract

We discuss when a unital homomorphism $\phi : C(X) \to A$ can be approximated by finite-dimensional homomorphisms, where X is a compact metric space and A is unital simple C*-algebra with tracial rank one. In this paper, we will give a necessary and sufficient condition.

1. Introduction

In the theory of C*-algebras, studying homomorphisms between two C*-algebras is of fundamental importance. As a simple step, but also important, we study homomorphisms from some commutative C*-algebra $C(X)$, where X is a compact metric space, into some simple C*-algebra. Among these homomorphisms, the ones defined by evaluation at some finite points in X are the most simple case, or equivalently, the ones with finite-dimensional range (we call them finite-dimensional homomorphisms). Now, it is natural to study their limits (in the point-wise convergence topology).

In this paper, the target C*-algebra of a homomorphism we shall consider is in an important class of simple C*-algebras in the classification theory, the unital simple C*-algebras with tracial rank no more than one. It is introduced by H. Lin to aid the program of classification of nuclear C*-algebras ([Lin2]). H. Lin completely classified the unital nuclear separable simple C*-algebras with tracial rank one which satisfy the UCT, see [Lin4]. Let A be a unital simple C*-algebra with tracial rank no more than one, consider a unital monomorphism $\phi : C(X) \to A$. (For this problem, we only need to consider monomorphism.) When X is path-connected and A is of tracial rank zero, it is proved that ϕ can be approximated by finite-dimensional homomorphisms if and only if ϕ induces an zero element in $KL(C_0(X), A)$ (see [HLX]). In the present paper, we shall extend the result to the tracial rank one case. It is worth mentioning that the latter C*-algebras are not of real rank zero. It turns out that ϕ can be approximated by finite-dimensional homomorphisms if and only if $[\phi]$ vanishes on $\underline{K}(C_0(X))$, and in addition, the induced maps $\widehat{\phi}$ maps $\mathrm{Aff}T(C(X))$ into $\overline{\rho_A(K_0(A))}$ and $\phi^{\dagger}$ is trivial. For a general compact metric space X (not necessarily path-connected nor a disjoint union of finitely many path-connected spaces), we need some new generalized notation to describe $[\phi]$ (see Definition 3.4).

In the literature, this problem is related to the properties such as real rank zero, (FU) and (FN), corresponding to $X = [0,1], \mathbb{T}$, and a compact subset in the complex plane, respectively ([Lin1] and [Lin8]). In [Lin8], it is shown that a unitary in a unital simple C*-algebra with tracial rank one can be approximated by unitaries with finite spectrum if and only if $u \in CU(A)$ and $\widehat{u^n + (u^n)^*}, i\widehat{(u^n - (u^n)^*)} \in \overline{\rho_A(K_0(A))}$ for all $n \geq 1$. As an application of the main result in this paper , we shall describe when a normal element in a unital simple C*-algebra with tracial rank one can be approximated by normal elements with finite spectrum.

2. Preliminaries

In this section, we gather some notations and well-known facts.

2.1. Let A, B be two C*-algebras and let $\phi, \psi : A \to B$ be two maps. Suppose that $\mathcal{F} \subset A$ and $\epsilon > 0$. We write

$$\phi \approx_\epsilon \psi \text{ on } \mathcal{F}$$

if $\| \phi(x) - \psi(x) \| < \epsilon$ for all $a \in \mathcal{F}$. Similarly, we write

$$\phi \approx_\epsilon \mathrm{ad}u \circ \psi \text{ on } \mathcal{F}$$

if there is a unitary $u \in B$ such that $\| \phi(x) - u\psi(x)u^* \| < \epsilon$ for all $a \in \mathcal{F}$.

2.2. If $X = X_1 \sqcup \cdots \sqcup X_m$ is a disjoint union of path-connected compact metric spaces with each component X_i a base point $x_i \in X_i$ for $i = 1, \cdots, m$. We shall use the notation $C_0(X)$ to mean the set of continuous functions on X which vanish at all x_i. Put $rC(X) = C(X)/C_0(X) \cong \mathbb{C}^m$ ([DN] and [EG]).

Let A be a unital C*-algebra and $\phi : C(X) \to A$ be a unital homomorphism. Then ϕ defines an element $[\phi]$ in $KK(C(X), A)$. It is known that the short exact sequence

$$0 \to C_0(X) \to C(X) \to rC(X) \to 0$$

is split and there is a natural decomposition,

$$KK(C(X), A) = KK(C_0(X), A) \oplus KK(rC(X), A).$$

If $\beta \in KK(C(X), A)$, we will write $\beta = (\beta_0, \beta_1)$ under this decomposition. In particular, suppose that $\phi : C(X) \to A$ is a unital homomorphism and denote by $e_i \in C(X)$ the identity of $C(X_i)$, then $[\phi] = ([\phi]_0, [p_1], \cdots, [p_m]) \in KK(C_0(X), A) \oplus KK(rC(X), A) = KK(C_0(X), A) \oplus K_0(A) \oplus \cdots \oplus K_0(A)$, where $p_i = \phi(e_i), i = 1, \cdots, m$.

From the universal coefficient theorem (see [RS] or [Bl]), for the C*-algebras $C = C(X)$ and A as above, there is a split short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(C), K_*(A)) \to KK(C, A) \xrightarrow{\gamma} \mathrm{Hom}^0(K_*(C), K_*(A)) \to 0.$$

Define

$$KL(C, A) = KK(C, A)/\mathrm{Pext}^1_{\mathbb{Z}}(K_*(C), K_*(A)).$$

2.3. Let A be a C*-algebra and let C_n be a commutative C*-algebra with $K_0(C_n) = \mathbb{Z}/n$ and $K_1(C_n) = 0$. We use the following notation:

$$K_*(A, \mathbb{Z}/n) = K_*(A \otimes C_n)$$

and

$$\underline{K}(A) = K_*(A) \oplus \bigoplus_{n=2}^{\infty} K_*(A, \mathbb{Z}/n).$$

Denote by $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ the set of systems of group homomorphisms which is compatible with all the Bockstein Operations (see [DL] for details). From [DL], we know that

$$KL(A, B) = Hom_\Lambda(\underline{K}(A), \underline{K}(B))$$

for A is a separable amenable C*-algebra which satisfies the UCT.

If $\alpha \in KL(C(X), B)$, we use the notation $\alpha = \{\alpha_n^i\}$, where $\alpha_n^i \in Hom(K_i(C(X), \mathbb{Z}/n) \to K_i(B, \mathbb{Z}/n)), i = 0, 1$.

2.4. Let X be a compact metric space and $x, y \in X$. Let $\delta > 0$. We write $x \sim_\delta y$, if there are points $x_0, x_1, \cdots, x_m$ in X such that

$$x_0 = x, x_m = y, \text{ and } dist(x_i, x_{i+1}) < \delta,$$

for $i = 0, 1, \cdots, m-1$. A subset $Y \subset X$ is said to be δ-connected, if for any two points x, y in Y, one has $x \sim_\delta y$.

It is well-known that for any $\delta > 0$, X can be divided into finitely many disjoint δ-connected components. We will frequently use the following lemma.

Lemma (Lemma 3.3 of [HLX]) *Let X be a compact metric space and $G \subset \underline{K}(C(X))$ be a finitely generated subgroup. Then there exists $\delta > 0$ satisfying the following:*

If $X = X_1 \sqcup \cdots \sqcup X_m$, where each X_i is δ-connected, A is a unital C^*-algebra, and $\phi, \psi : C(X) \to A$ are two unital finite-dimensional homomorphisms such that $[\phi]([e_i]) = [\psi]([e_i])$ for $i = 1, \cdots, m$, then

$$[\phi] \mid_G = [\psi] \mid_G .$$

□

2.5. For a unital C*-algebra A, let $T(A)$ denote the space of all tracial states of A. It is well known that $T(A)$ is a Choquet simplex. Let $\mathrm{Aff}T(A)$ be the space of all affine continuous real functions on $T(A)$. Then $\mathrm{Aff}T(A)$ is an ordered Banach space with order unit. If X is a compact Hausdorff space, then it is well known that $\mathrm{Aff}T(C(X)) = C_{\mathbb{R}}(X)$, the space of all real continuous functions on X.

2.6. Let $\phi : C \to A$ be a unital homomorphism. Denote by $\phi^T : T(A) \to T(C)$ the affine continuous map induced by ϕ, i.e. $\phi^T(\tau) = \tau \circ \phi$ for all τ in $T(A)$. It then also induces a unital positive linear map $\widehat{\phi} : \mathrm{Aff}T(C) \to \mathrm{Aff}T(A)$ defined by $\widehat{\phi}(f) = f \circ \phi^T$ for all f in $\mathrm{Aff}T(C)$.

Let A be a unital C*-algebra, denote by $(K_0(A), K_0(A)^+, [1_A])$ the associated scaled ordered group and $SK_0(A)$ the state space of $K_0(A)$ (see [R]). There is an affine continuous map $r_A : T(A) \to SK_0(A)$ defined by $r_A(\tau)([p]) = \sum_{i=1}^n \tau(p_{ii})$, where $\tau \in T(A)$ and $[p] \in K_0(A)$ is the element presented by the projection $p \in M_n(A)$. Then r_A defines a canonical map $\rho_A : K_0(A) \to \mathrm{Aff}T(A)$ by $\rho_A(g)(\tau) = r_A(\tau)(g)$ for τ in $T(A)$ and g in $K_0(A)$.

Let $\pi_A : \mathrm{Aff}T(A) \to \mathrm{Aff}T(A)/\overline{\rho_A(K_0(A))}$ denote the canonical quotient map.

2.7. Let $U(A)$ be the unitary group of A, and $CU(A)$ the closure of the commutator subgroup of $U(A)$. Using de la Harpe-Skandalis determinant, by Theorem 3.2 of [Th], one has the following splitting short exact sequence:

$$0 \to \mathrm{Aff}T(A)/\overline{\rho_A(K_0(A))} \to U_\infty(A)/CU_\infty(A) \to K_1(A) \to 0. \tag{2.1}$$

We then have

$$U_\infty(A)/CU_\infty(A) \cong \mathrm{Aff}T(A)/\overline{\rho_A(K_0(A))} \oplus K_1(A). \tag{2.2}$$

For a unital homomorphism $\phi : C \to A$, it induces a group homomorphism $\phi^\ddagger : U_\infty(C)/CU_\infty(C) \to U_\infty(A)/CU_\infty(A)$. With respective to the decomposition (2.2), we can view $\phi^{\ddagger}$ as a matrix:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

Here α_{21} is automatically zero by exactness, α_{11} is induced by $\widehat{\phi}$, and α_{22} is the map ϕ_{*1}. Denote the rest homomorphism α_{12} by

$$\phi^{\dagger} : K_1(C) \to \mathrm{Aff}T(A)/\overline{\rho_A(K_0(A))}.$$

2.8. We shall recall the definition and some basic properties of unital simple C*-algebras of tracial ranks (see [Lin2] and [Lin4]).

Let A be a unital simple C*-algebra and $k \in \mathbb{N}$. We say that A has **tracial rank no more than** k if for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$ and any non-zero positive element $a \in A_+$, there exist a non-zero projection $p \in A$ and a C*-subalgebra B of A with $1_B = p$ such that

(0) B has form $B = \oplus_{i=1}^{q} P_{n_i} M_{n_i}(C(X_i)) P_{n_i}$, where P_{n_i} are projections in $M_{n_i}(C(X_i))$ and X_i is a finite CW complex with $dim(X_i) \leq k$ for each i,

(1) $\| px - xp \| < \epsilon$ for all $x \in \mathcal{F}$,

(2) $pxp \in_{\epsilon} B$ for all $x \in \mathcal{F}$,

(3) $[1 - p] \leq [a]$.

We will write $TR(A) \leq k$ if A has tracial rank no more than k. Especially, $TR(A) = 0$ means that A has tracial rank zero, i.e. the above B can be chosen to be finite dimensional.

Recently, H. Lin proved that if a unital simple C*-algebra A with $TR(A) \leq k$ satisfies the UCT, then A actually has tracial rank no more that one (see [Lin11]). Hence we focus on A with $TR(A) \leq 1$.

Suppose that A is a unital simple C*-algebra with $TR(A) \leq 1$, then it is well known that A has stable rank one, real rank no more than one, weakly unperforated K_0-group with Riesz interpolation property and fundamental comparison property (see [Lin4]). For $TR(A) = 0$ case, we know that A has real rank zero and the canonical map $\rho_A : K_0(A) \to \mathrm{Aff}T(A)$ in 2.6 has dense range when A is infinite dimensional simple C*-algebra.

3. Main Results

First of all, we discuss the case when X is path-connected. The following result is the main theorem in [HLX].

Lemma 3.1. (Theorem 3.9 of [HLX]) *Let X be a compact path-connected metric space, and A be a unital simple C*-algebra with $TR(A) = 0$. Suppose* *that $\phi : C(X) \to A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$[\phi]_0 = 0 \text{ in } KL(C_0(X), A).$$

□

By using the uniqueness theorem recently proved by H. Lin ([Lin10]) and the method used in [Lin8], we obtain the following result.

Theorem 3.2. *Let X be a compact path-connected metric space with a based point x_0, and A be a unital simple infinite dimensional C^*-algebra with $TR(A) \le 1$. Suppose $\phi : C(X) \to A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$[\phi]_0 = 0 \text{ in } KL(C_0(X), A),$$

$$\pi_A \circ \hat{\phi} = 0, \text{ and}$$
$$\phi^\dagger = 0.$$

Proof. Firstly, suppose that $\psi : C(X) \to A$ is a unital finite-dimensional homomorphism. Then we can write ψ as

$$\psi(f) = \sum_{k=1}^{m} f(x_k) p_k$$

for all $f \in C(X)$, where $x_k \in X$ and $p_1, \ldots, p_m$ are mutually orthogonal projections in A with $\sum_{k=1}^m p_k = 1$. Define $\psi_0 : C(X) \to A$ by

$$\psi_0(f) = f(x_0) \cdot 1_A$$

for all $f \in C(X)$. Since X is path-connected, then ψ is homotopic to ψ_0, and hence

$$[\psi]_0 = [\psi_0]_0 = 0, \text{ in } KL(C_0(X), A).$$

Also, since $\overline{\rho(K_0 A)}$ is a $\mathbb{R}$-linear subspace of $\mathrm{Aff}T(A)$ (see Proposition 3.6 of [Lin8]), we see that $\widehat{\psi}$ maps $C_{\mathbb{R}}(X)$ into $\overline{\rho(K_0 A)}$. Next, if $u \in U(C(X))$, then $| u(x_k) | = 1$, we write $u(x_k) = exp(i\theta_k)$, where $\theta_k \in \mathbb{R}$ for $k = 1, \cdots, m$. Put $h = \sum_{k=1}^m \theta_k p_k \in A_{sa}$, then

$$\psi(u) = \sum_{k=1}^{m} u(x_k) p_k = exp(ih).$$

Note that $\widehat{h} \in \overline{\rho(K_0 A)}$, $\psi(u) \in CU(A)$ by Theorem 2.9 of [Lin8]. Hence $\psi^\ddagger(U(C(X))) \subset CU(A)$. Similarly, $\psi^\ddagger(U_n(C(X))) \subset CU_n(A)$. Then $\psi^\ddagger =$

0, and hence $\psi^\dagger = 0$. Now if ϕ can be approximated by finite-dimensional homomorphisms, ϕ must satisfies the mentioning three conditions.

Conversely, suppose that $\phi : C(X) \to A$ is a unital homomorphism such that $[\phi]_0 = 0$, $\pi_A \circ \widehat{\phi} = 0$, and $\phi^\dagger = 0$. As the proof of Lemma 4.1 of [Lin8], we can choose a unital simple C*-subalgebra $B \subset A$ with tracial rank zero such that the inclusion $\imath : B \to A$ induces an isomorphism:

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) \cong (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Then $[\imath]$ is a KK-equivalence in $KK(B, A)$ (see 7.6 of [RS]), and hence there is a $\beta \in KK(A, B)$ such that

$$[\imath] \times \beta = [id_B] \text{ and } \beta \times [\imath] = [id_A].$$

Define $\kappa = [\phi] \times \beta \in KL_e(C(X), B)^{++}$. Since $TR(B) = 0$, we know that $\mathrm{Aff}T(B) = \overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$. By assumption, $\widehat{\phi}$ maps $C_{\mathbb{R}}(X)$ into $\overline{\rho_A(K_0(A))}$, then $\widehat{\phi}$ defines a unital strickly positive linear map $\gamma : C_{\mathbb{R}}(X) \to \mathrm{Aff}T(B)$ such that $\widehat{\imath} \circ \gamma = \widehat{\phi}$. We now check that the defined pair (κ, γ) is compatible, that is, $\rho_B \circ \kappa_0^0 = \gamma \circ \rho_{C(X)}$. It suffices to show that $\widehat{\imath} \circ \rho_B \circ \kappa_0^0 = \widehat{\imath} \circ \gamma \circ \rho_{C(X)}$, since $\widehat{\imath}$ is injective. This is equivalent to

$$\rho_A \circ \widehat{\imath} \circ \kappa_0^0 = \widehat{\phi} \circ \rho_{C(X)}. \tag{3.1}$$

Since $\widehat{\imath} \circ \kappa_0^0 = \phi_{*0}$, the equation (3.1) becomes

$$\rho_A \circ \phi_{*0} = \widehat{\phi} \circ \rho_{C(X)},$$

and this is well-known true. The following diagram shows the above calculation:

$$\begin{array}{ccccc} K_0(C(X)) & \xrightarrow{\kappa_0^0} & K_0(B) & \underset{\imath_{*0}}{\overset{\beta_0^0}{\rightleftarrows}} & K_0(A) \\ \downarrow{\scriptstyle \rho_{C(X)}} & & \downarrow{\scriptstyle \rho_B} & & \downarrow{\scriptstyle \rho_A} \\ C_{\mathbb{R}}(X) & \xrightarrow[\gamma]{} & \mathrm{Aff}T(B)) & \xrightarrow[\widehat{\imath}]{} & \mathrm{Aff}T(A) \end{array}$$

Now apply Theorem 5.2 of [Lin6], there exists a unital monomorphism $h : C(X) \to B$ such that

$$[h] = \kappa \text{ in } KL(C(X), B) \quad \text{and } \widehat{h} = \gamma. \tag{3.2}$$

Then

$$[\imath \circ h] = [h] \times [\imath] = \kappa \times [\imath] = [\phi] \times \beta \times [\imath] = [\phi], \tag{3.3}$$

and

$$\widehat{(\imath \circ h)} = \widehat{\imath} \circ \widehat{h} = \widehat{\imath} \circ \gamma = \widehat{\phi}. \tag{3.4}$$

Note that $TR(B) = 0$, $h^\dagger$ is automatically zero and then

$$(\imath \circ h)^\dagger = \phi^\dagger = 0. \tag{3.5}$$

Combine (3.3) – (3.5) and use Theorem 5.10 of [Lin10], we conclude that the two homomorphisms ϕ and $\imath \circ h$ are approximately unitarily equivalent. Finally, since $[\phi]_0 = 0$ in $KL(C_0(X), A)$, it is obviously that $[h]_0 = 0$ in $KL(C_0(X)), B)$. From Lemma 3.1, h can be approximated by finite-dimensional homomorphisms, and so is ϕ. □

Corollary 3.3. *Let X be a disjoint union of finitely many compact path-connected metric spaces, and A be a unital simple infinite dimensional C*-algebra with $TR(A) \leq 1$. Suppose $\phi : C(X) \to A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$[\phi]_0 = 0 \text{ in } KL(C_0(X), A),$$
$$\pi_A \circ \hat{\phi} = 0, \text{ and}$$
$$\phi^\dagger = 0.$$

Proof. Write ϕ as a finite direct sum and apply Theorem 3.2. □

Now we turn to consider a general (not necessarily path-connected) compact metric space X. We need a condition which generalize the first condition of Corollary 3.3. For any $x \in X$, the evaluation map $\pi_x : C(X) \to \mathbb{C}$ defines an element $[\pi_x] : \underline{K}(C(X)) \to \underline{K}(\mathbb{C})$.

Let A be a unital C*-algebra and $\phi : C(X) \to A$ be a unital homomorphism. We shall use the following equation

$$[\phi] \mid_{\cap_{x\in X} Ker[\pi_x]} = 0, \tag{3.6}$$

to describe ϕ. It is easy to see that (3.6) is equivalent to say $[\phi]_0 = 0$ when X is a disjoint union of finitely many path-connected compact spaces.

Let X be a compact metric space. It is well-known that there is an inductive system $\{C(X_n), h_n\}$ such that $C(X) = \varinjlim(C(X_n), h_n)$, where each X_n is a finite CW complex. Suppose that $h_{n,\infty} : C(X_n) \to C(X)$ is the induced homomorphism. Also there are continuous maps $f_n : X \to X_n$ induce $h_{n,\infty}$. Write $X_n = X_n^1 \sqcup \cdots \sqcup X_n^{r(n)}$, where X_n^i is the path-connected component of X_n for $i = 1, \cdots, r(n)$. Fix a point x_n^i in each X_n^i.

Lemma 3.4. *With above notations, let A be a unital C*-algebra and ϕ : $C(X) \to A$ be a unital homomorphism. If (3.6) holds, then we have*

$$[\phi \circ h_{n,\infty}] \mid_{\cap_{i=1}^{r(n)} ker[\pi_{x_n^i}]} = 0, \tag{3.7}$$

for all n.

Proof. Fix a number n. For any $g \in \cap_{i=1}^{r(n)} ker[\pi_{x_n^i}]$, let G be the subgroup of $\underline{K}(C(X))$ generated by $[h_{n,\infty}](g)$. Let $\delta > 0$ be as required by Lemma 2.4 for G. Write $X = Y_1 \sqcup \cdots \sqcup Y_N$, where each Y_k is a δ-connected component of X. For each $1 \le k \le N$, there is $1 \le i(n,k) \le r(n)$ such that $f_n(Y_k) \cap X_n^{i(n,k)} \ne \emptyset$. Choose points $y_k \in Y_k$ and $z_n^{i(k)} \in X_n^{i(n,k)}$ such that

$$f_n(y_k) = z_n^{i(k)},$$

for each k. Then

$$[\pi_{y_k}]([h_{n,\infty}](g)) = [\pi_{z_n^{i(k)}}](g) = [\pi_{x_n^{i(k)}}](g) = 0. \tag{3.8}$$

By Lemma 2.4, we obtain that $[\pi_y]([h_{n,\infty}](g)) = 0$ for all $y \in Y_k$. This is true for each k and then for all $y \in X$. Hence, (3.7) is now followed from (3.6). □

The key point of using equation (3.6) is shown in the next lemma.

Lemma 3.5. *Let $\phi : C(X) \to A$ be a unital homomorphism as above, then the following two conditions are equivalent:*

(i) $[\phi] |_{\cap_{x\in X} Ker[\pi_x]} = 0$;

(ii) *for any finitely generated subgroup $G \subset \underline{K}(C(X))$, there is a unital finite dimensional homomorphism $\psi : C(X) \to A$ such that*

$$[\phi] |_G = [\psi] |_G .$$

Proof. Firstly, we prove "(i) ⇒ (ii)". Suppose that $C(X) = \varinjlim (C(X_n), h_n)$, where each X_n is a finite CW complex. Write $X_n = X_n^1 \sqcup \cdots \sqcup X_n^{r(n)}$, where X_n^i is the path-connected component of X_n for $i = 1, \cdots, r(n)$. And denote by e_n^i the unit of the summand $C(X_n^i)$. Then, we can further assume that h_n is unital and $h_n(e_n^i) \ne 0$ for all n, i, since otherwise, we can delete the summand $C(X_n^i)$ from the inductive system without changing the inductive limit. Suppose that $h_{n,\infty} : C(X_n) \to C(X)$ is the induced map. It follows that

$$h_{n,\infty}(e_n^i) \ne 0 \tag{3.9}$$

for all n, i. On the other hand, there are continuous maps $f_n : X \to X_n$ induce $h_{n,\infty}$. Then (3.9) is equivalent to

$$f_n(X) \cap X_n^i \ne \emptyset \tag{3.10}$$

for $i = 1, \cdots, r(n)$.

Given a finitely generated subgroup $G \leq \underline{K}(C(X))$, we can choose n large enough such that $G \subset [h_{n,\infty}](\underline{K}(C(X_n)))$. From (3.10), we can choose $x_n^i \in X_n^i$ and $y^i \in X$ such that

$$f_n(y^i) = x_n^i \tag{3.11}$$

for each $i = 1, \cdots, r(n)$. Now, consider $\phi \circ h_{n,\infty} : C(X_n) \to A$. By Lemma 3.4, we have

$$[\phi \circ h_{n,\infty}] \mid_{\cap_{i=1}^{r(n)} ker[\pi_{x_n^i}]} = 0. \tag{3.12}$$

Define $\psi : C(X) \to A$ by

$$\psi(f) = \sum_{i=1}^{r(n)} f(y^i) p_i,$$

where $p_i = \phi \circ h_{n,\infty}(e_n^i)$. From (3.12) and the fact that the homomorphisms are unital, we obtain that $[\phi \circ h_{n,\infty}] = [\psi \circ h_{n,\infty}]$, and hence $[\phi] \mid_G = [\psi] \mid_G$.

Conversely, suppose that the condition (ii) is true. Let g be an element in $\cap_{x\in X} ker[\pi_x]$. Let G be the group generated by g. Now we can choose a unital finite dimensional homomorphism $\psi : C(X) \to A$ such that

$$[\phi] \mid_G = [\psi]_G. \tag{3.13}$$

Choose a large n satisfying the conditions as above. Then there is $\widetilde{g} \in \underline{K}(C(X_n))$ such that $g = [h_{n,\infty}](\widetilde{g})$. Since $g \in \cap_{x\in X} Ker[\pi_x]$, by (3.11) , we have $\widetilde{g} \in \cap_{i=1}^{r(n)} ker[\pi_{x_n^i}]$. Therefore by (3.13),

$$[\phi](g) = [\phi] \circ [h_{n,\infty}](\widetilde{g}) = [\psi] \circ [h_{n,\infty}](\widetilde{g}) = 0.$$

□

Lemma 3.6. *Let A be a unital C*-algebra satisfying the fundamental comparison property, $\phi : C(X) \to A$ be a unital homomorphism. If $\widehat{\phi}(C_{\mathbb{R}}(X)) \subset \overline{\rho_A(K_0(A))}$, then for any finite subset $\mathcal{F} \subset C_{\mathbb{R}}(X)$ and $\epsilon > 0$, there exists a unital finite dimensional homomorphism $\psi : C(X) \to A$ such that*

$$\widehat{\phi} \approx_\epsilon \widehat{\psi} \text{ on } \mathcal{F}.$$

Proof. The proof is contained in the proof of Theorem 3.6 of [HLX]. □

For our purpose, we need further the following approximated version of the uniqueness theorem in [Lin10].

Lemma 3.7. (Theorem 5.3 of [Lin10]) L*et X be a compact metric spaces, and A be a unital simple C*-algebra with $TR(A) \leq 1$. Suppose $\phi : C(X) \to$*

A is a unital monomorphism. For any finite subset $\mathcal{F} \subset C(X)$ and $\epsilon > 0$, there exist a finitely generated subgroup $G \subset \underline{K}(C(X))$, a finite subset $\mathcal{G} \subset C_{\mathbb{R}}(X)$ and $\gamma > 0$ such that

For any unital homomorphism $\psi : C(X) \to A$, if

$$[\phi] \mid_G = [\psi]_G,$$

$$\widehat{\phi} \approx_\gamma \widehat{\psi} \text{ on } \mathcal{G},$$

and

$$\phi^{\ddagger} = \psi^{\ddagger},$$

then there is a unitary $u \in A$ such that

$$\phi \approx_\epsilon \mathrm{ad}u \circ \psi \quad on \quad \mathcal{F}. \qquad \square$$

With the above preparations, now we can prove the main lemma.

Lemma 3.8. *Let X be a compact metric spaces, and A be a unital simple infinite dimensional C*-algebra with $TR(A) \leq 1$. Suppose $\phi : C(X) \to A$ is a unital monomorphism satisfying the following conditions:*

(i) *for any finitely generated subgroup $G \subset \underline{K}(C(X))$, there is a unital finite dimensional homomorphism $\psi : C(X) \to A$ such that*

$$[\phi] \mid_G = [\psi] \mid_G,$$

(ii)

$$\phi^{\ddagger} = 0.$$

Then ϕ can be approximated by finite-dimensional homomorphisms.

Proof. It suffices to prove the following: for any finite subset $\mathcal{F} \subset C(X)$ and $\epsilon > 0$, there is a unital finite dimensional homomorphism $\psi : C(X) \to A$ and a unitary $u \in A$ such that

$$\phi \approx_\epsilon \mathrm{ad}u \circ \psi \quad on \quad \mathcal{F}.$$

Let $G \subset \underline{K}(C(X))$ be a finitely generated subgroup, $\mathcal{G} \subset C_{\mathbb{R}}(X)$ a finite subset and $\gamma > 0$ with the properties described in Lemma 3.7 corresponding to $\mathcal{F}$ and ϵ. For this finitely generated subgroup G, we can choose $\delta > 0$ as in Lemma 2.4. Write $X = X_1 \sqcup \cdots \sqcup X_m$, where each X_k is a δ-connected component of X.

Now let G_0 be the group generated by G and $\{[e_0], \cdots, [e_m]\}$. From the condition (i) we can choose a unital finite dimensional homomorphism $\psi_0 : C(X) \to A$ such that

$$[\phi] \mid_{G_0} = [\psi_0] \mid_{G_0} . \qquad (3.14)$$

Under the identification $C(X) = C(X_1) \oplus \cdots \oplus C(X_m)$, we consider the homomorphism $\phi_i : C(X_i) \to p_i A p_i$ induced by ϕ, where $p_i = \phi(e_i)$ for $i = 1, \cdots, m$. Also view $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$. Since $\phi^{\ddagger} = 0$, we have

$$\hat{\phi}(C_{\mathbb{R}}(X)) \subset \overline{\rho_A(K_0(A))}. \tag{3.15}$$

And then we also have $\hat{\phi}_i(C_{\mathbb{R}}(X_i)) \subset \overline{\rho_{p_iAp_i}(K_0(p_iAp_i))}$. For $\mathcal{G}_i$ and $\gamma > 0$, by applying Lemma 3.6, there is a unital finite-dimensional homomorphism $\phi_i^{'} : C(X_i) \to p_i A p_i$, such that

$$\widehat{\phi_i} \approx_{\gamma} \widehat{\phi_i^{'}} \text{ on } \mathcal{G}. \tag{3.16}$$

Define $\psi : C(X) \to A$ by

$$\psi = \phi_1^{'} \oplus \cdots \oplus \phi_m^{'}.$$

It follows from Lemma 2.4, (3.14), and the definition of ψ that

$$[\psi] \mid_G = [\psi_0] \mid_G = [\phi] \mid_G . \tag{3.17}$$

From (3.16), we have

$$\widehat{\phi} \approx_{\gamma} \widehat{\psi} \text{ on } \mathcal{G}. \tag{3.18}$$

Note that $\phi^{\ddagger} = \psi^{\ddagger} = 0$. We complete the proof by applying Lemma 3.7. □

Theorem 3.9. *Let X be compact metric spaces, and A be a unital simple infinite dimensional C^*-algebra with $TR(A) \leq 1$. Suppose $\phi : C(X) \to A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$[\phi] \mid_{\cap_{x \in X} Ker[\pi_x]} = 0,$$

$$\pi_A \circ \hat{\phi} = 0, \text{ and}$$

$$\phi^{\dagger} = 0.$$

Proof. For the necessity, we need only to check the first condition. This is easy by Lemma 3.5. Conversely, combine Lemma 3.5 and 3.8, and also note that the three conditions imply that $\phi^{\ddagger} = 0$, see 2.7. □

As an application, we now obtain a necessary and sufficient condition for a normal element being approximated by normal elements with finite spectrum.

Corollary 3.10. *Let A be a unital simple infinite dimensional C^*-algebra with $TR(A) \leq 1$ and $x \in A$ be a normal element. Suppose that ϕ :*

$C(sp(x)) \to A$ is the map induced by continuous functional calculus. Then x can be approximated by normal elements with finite spectrum if and only if

$$\phi^{\ddagger} = 0.$$

Proof. Put $X = sp(x)$. We know that $K_*(C(X))$ is torsion free, then

$$KL(C(X), A) = \bigoplus_{i=0,1} Hom(K_i(C(X)), K_i(A)).$$

Note that in this case, $[\phi]_0 = 0$ is equivalent to say that $\phi_{*1} = 0$. Then the three conditions in Theorem 3.9 follow by $\phi^{\ddagger} = 0$. □

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